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# The Method of Boundary Perturbation, and Its Application to Wakefield Calculations†

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## Abstract

The boundary perturbation method, suggested by Zhang and, independently, by Chatard-Moulin, Cooper, and their colleagues, is employed to the wakefield calculations for geometrical discontinuities in accelerators. The results are compared with that obtained from the mesh caculations using TBCI. [1] When the perturbation is small and the geometry is suitable for TBCI, the agreement is good. The discrepancies observed in other cases are also discussed.


## 1. Introduction

It is known that wakefields play an important role in designing modern large accelerators. There are several existing methods to estimate wakefields (or their counterparts in the frequency-domain, impedances), each with its own limitations. [2] The analytic formulae, either in a closed form or expressed as a series expansion obtained from the field matching method, work only for a few special simple cases. The numerical method, e. g., mesh calcualtions using TBCI, works generally well, but troubles have been experienced in certain cases, in particular for long structures. As an alternative approach, this paper implements the method of boundary perturbation (BP) to computing the wakefields generated by a Gaussian bunch.[1] This method is developed by Zhang [3, 4] and, independently, by Chatard-Moulin [5], Cooper [6], and their colleagues. Zhang introduces a general format, whereas Chatard-Moulin and Cooper consider a special case that is most relevant to accelerators.

## 2. The Method of Boundary Perturbation

The basic idea of Zhang's method is to transform the boundary conditions on the *perturbed* boundaries (which are given) to that on the *unperturbed* boundaries (which are not known), so that the differential equations would then be easier to solve.

Let $z$ and $r$ be the longitudinal and radial coordinates, respectively. Assume $b_0(z)$ be an unperturbed smooth boundary, and $b(z)$ the perturbed one. Let

$$\epsilon h(z) = b(z) - b_0(z) \ , \tag{1}$$

[1]The wakefields of a bunch are sometimes called the *wake potentials* to be distinguished from that of a point charged particle.

in which $\epsilon$ is a small real number (the perturbation parameter), see Fig. 1. Assume that $f$ is a solution (to be found) of some differential equation defined in the $(r, z)$ plane:

$$\mathbf{D}f = 0 \ , \tag{2}$$

and that its value on the perturbed boundary, $f(b)$, is given. In view of the standard perturbation technique, we express $f$ in terms of a series of functions:

$$f = f^{(0)} + \epsilon f^{(1)} + \epsilon^2 f^{(2)} + \ldots \ , \tag{3}$$

in which $f^{(i)}$s $(i = 0, 1, 2\ldots)$ are the solutions of Eq. (2) with proper values on the *unperturbed* boundary $b_0$, which are to be discussed below.

The Taylor series of $f(b)$ is

$$f(b) = f(b_0) + (\epsilon h)\frac{\partial f}{\partial r}(b_0) + \frac{(\epsilon h)^2}{2}\frac{\partial^2 f}{\partial r^2}(b_0) + \ldots \tag{4}$$

This can be rewritten as

$$f(b_0) = f(b) - (\epsilon h)\frac{\partial f}{\partial r}(b_0) - \frac{(\epsilon h)^2}{2}\frac{\partial^2 f}{\partial r^2}(b_0) - \ldots \tag{5}$$

Plugging (3) into (5) and collecting terms, we get

$$\begin{aligned} f^{(0)}(b_0) + \epsilon f^{(1)}(b_0) + \epsilon^2 f^{(2)}(b_0) + \ldots = f(b) \ & - \ \epsilon h \frac{\partial f^{(0)}}{\partial r}(b_0) \\ & - \ \epsilon^2 [h \frac{\partial f^{(1)}}{\partial r}(b_0) + \frac{h^2}{2}\frac{\partial^2 f^{(0)}}{\partial r^2}(b_0)] \\ & - \ \ldots \end{aligned} \tag{6}$$

Thus, we may obtain the conditions on the *unperturbed* boundary $b_0$ by the following procedure:

$$\begin{aligned} f^{(0)}(b_0) &= f(b) \\ f^{(1)}(b_0) &= -h\frac{\partial f^{(0)}}{\partial r}(b_0) \\ f^{(2)}(b_0) &= -[h\frac{\partial f^{(1)}}{\partial r}(b_0) + \frac{h^2}{2}\frac{\partial^2 f^{(0)}}{\partial r^2}(b_0)] \\ &\ldots \\ f^{(i)}(b_0) &= -\sum_{q=1}^{i} \frac{h^q}{q!}\frac{\partial^q f^{(i-q)}}{\partial r^q}(b_0) \end{aligned} \tag{7}$$

Once $f^{(i)}$s are solved for these conditions, $f$ can be obtained from Eq. (3). Strictly speaking, Eq. (3) is insignificant unless we can prove that

- The perturbation series converge and the limit is exactly $f$, and
- $f^{(1)}$ is the 1st order perturbation, $f^{(2)}$ the 2nd order, etc.

A general proof is difficult. For several specific differential equations and boundary conditions, the proof can be found in [3, 4].

This perturbation technique can be easily extended to the case in which the boundaries are multi-dimensional. The generalized Eq. (5) takes the form

$$f(\vec{b}_0) = f(\vec{b}) - (\epsilon\vec{h})\frac{\partial f}{\partial \vec{r}}(\vec{b}_0) - \frac{(\epsilon\vec{h})^2}{2}\frac{\partial^2 f}{\partial \vec{r}^2}(\vec{b}_0) + \ldots \tag{8}$$

And Eq. (7) will then be modified accordingly.

In their studies on beam dynamics in accelerators, Chatard-Moulin, Cooper and their colleagues employ basically the same technique to a specific case and get the solutions up to the 2nd order. They show that, if the following conditions are satisfied,

(a) The geometry is rotationally symmetric,

(b) The wall is superconducting, and

(c) The boundary is periodically perturbed,

then $f^{(0)}$, $f^{(1)}$, and $f^{(2)}$ ($f$ stands for either the electric field $\vec{E}$ or magnetic field $\vec{B}$) are indeed solvable. [5, 6]

## 3. Application to Wakefield Calculations

Let the Fourier series of a periodic structure with period $L$ be

$$b(z) = b_0[1 + \sum_{p=-\infty}^{+\infty} c_p \cdot e^{j(2\pi p/L)z}] \ , \tag{9}$$

with $c_0 = 0$. Assume a Gaussian bunch of $rms$ length $\sigma$ traversing this peoridic structure with the speed of light. Then one can show that the Chatard-Moulin and Cooper approach gives the following longitudinal and transverse wakefields per period:

$$\begin{aligned} W_{\|}(s)^{m=0}(V/pC) &= -1.8\pi \sum_{p=1}^{\infty} p \mid 2c_p \mid^2 \\ &\cdot \sum_{n=1}^{\infty} k_{0n} \, Re[\frac{1}{2}e^{-s^2/2\sigma^2} w(\frac{k_{0n}\sigma}{\sqrt{2}} - j\frac{s}{\sqrt{2}\sigma})] \end{aligned} \tag{10}$$

$$\begin{aligned} W_{\perp}(s)^{m=1}(V/pC \cdot m) &= -\frac{360\pi}{b_0^2} \sum_{p=1}^{\infty} p \mid 2c_p \mid^2 \\ &\cdot \sum_{n=1}^{\infty} \{ \frac{1}{1 - {x'_{1n}}^2} \, Im[\frac{1}{2}e^{-s^2/2\sigma^2} w(\frac{k'_{1n}\sigma}{\sqrt{2}} - j\frac{s}{\sqrt{2}\sigma})] \\ &- Im[\frac{1}{2}e^{-s^2/2\sigma^2} w(\frac{k_{1n}\sigma}{\sqrt{2}} - j\frac{s}{\sqrt{2}\sigma})] \} \end{aligned} \tag{11}$$

in which

$$k_{mn} = \frac{\pi p}{L} + \frac{L x_{mn}^2}{4\pi p b_0^2} \ , \qquad (12)$$

$$k'_{mn} = \frac{\pi p}{L} + \frac{L {x'_{mn}}^2}{4\pi p b_0^2} \ , \qquad (13)$$

$x_{mn}$ and $x'_{mn}$ are the $n^{th}$ root of the Bessel functions $J_m$ and $J'_m$, respectively, and $w$ is the complex error function. All lengths are in meters.

When the wakefields of a bunch are known, the effective impedance seen by the bunch can be readily obtained from a Fourier transform. This impedance spectra can then be converted to that associated with a point charge (the Green function) through a deconvolution. [7]

## 4. Comparison with Mesh Calculations Using TBCI

An interesting example is the transitions between beam chamber and insertion device (ID) section, see Fig. 2. The big tube is the regular beam chamber, while the small one represents the vacuum chamber in the ID sections (undulators and/or wigglers). They are connected to each other by tapered parts. This kind of structure is typical for the so-called *third generation of synchrotron radiation sources*, which usually accomodates a number of IDs. For this periodic structure, the Fourier coefficients are

$$c_p = a_p + j b_p \ , \qquad (14)$$

with

$$\begin{aligned} a_p &= 0 \\ b_p &= -\frac{2\epsilon}{\pi b_0} \cdot \frac{\sin(p\frac{\pi g}{L})}{\frac{\pi g}{L}} \cdot \frac{1}{p^2} \qquad for\ p = \pm 1, \pm 3, ... \\ &= 0 \qquad\qquad\qquad\qquad\qquad otherwise, \end{aligned} \qquad (15)$$

in which $b_0$ is the average radius, $\epsilon$ the maximum variation in the radial direction, and $g$ the length of the tapered part. These parameters are chosen in the test runs as follows:

$$\begin{aligned} b_0 &= 1.8\ cm, \\ \epsilon &= 0.2\ cm, \\ g &= 0.4\ cm, \\ L &= 8.0\ cm. \end{aligned} \qquad (16)$$

These choices are based on the following considerations:

(a) In order to get reliable results from BP, the radial variation should be small (in this example $\epsilon/b_0 \approx 0.1$).

(b) In order to match the boundary in TBCI calculation, the tapered angle $\theta$ should be either $90°$ or $45°$. The latter is obviously better so far as the perturbation method is concerned.

The longitudinal and transverse wakes for a Gaussian bunch of $\sigma = 1.75$cm passing this structure are shown in Figs. 3(a)-(b). In the range $[-5\sigma, 2\sigma]$, BP and TBCI give almost identical results. The discrepancies seen near the bunch tail may be attributed to the accumulation of numerical errors in TBCI.

However, when the conditions (a) and/or (b) above are violated, the results will be quite different. Let us take the 7-GeV APS storage ring as an example. The parameters of this machine are as follows:

$$\begin{aligned} b_0 &= 1.2 \ cm, \\ \epsilon &= 0.8 \ cm, \\ g &= 20 \ cm. \end{aligned} \tag{17}$$

The period length $L$ is $27.4\ m$. But the value of $L$ can be chosen as small as $50\, cm$ without affecting the calculated results, because of the so-called *composition rule* discussed in [8]. In this structure, the perturbation is large ($\epsilon/b_0 \approx 0.7$). This implies that the errors in the results obtained from BP could be substantial. On the other hand, the tapered angle $\theta$ is just about $5°$. This means that the matching between the TBCI-generated boundary and the real one is poor. Therefore, the wakes computed by the two methods in this case could differ from each other in a much more noticeable way. As illustrated in Fig. 4, the peak value of the transverse wake obtained from TBCI is about two times larger than that from BP.

Table 1 is a list of the longitudinal and transverse loss factors for different geometries, calculated by both BP and TBCI. When the perturbation is small and the tapered angle $\theta$ is $45°$, both methods give close results, as shown in No. 1, 2 and 3. In No. 4, the perturbation becomes large while $\theta$ remains to be $45°$. It seems that BP gives underestimated values while TBCI results are more believable. When the value of $\theta$ is decreased to $5°$, we see a very slow convergence in TBCI output, as discussed in [2]. Therefore, the listed values of the loss factors obtained from TBCI in No. 5 and 6 are overestimated. The actual values should be somewhere in between the BP and TBCI results.

## 5. Conclusions

We have demonstrated that the method of boundary perturbation is a valuable tool for computing the wakefields. When the perturbation is small, it gives reasonable results. Unlike the field matching method, it can be applied to various types of discontinuities, such as shielded bellows, weldments, valves, shielded end conflats, etc. In addition, this approach should be useful for proving certain analytical properties about the wakefields, loss factors and impedances. For example, we have found in our numerical simulations a power law that can describe the dependence of the transverse loss on the tapered angle for the structure shown in Fig. 2. [9] It is plausible to provide a proof by means of Eq. (11). This work is in plan.

# References

[1] TBCI is a computer code developed at DESY, West Germany.

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Table 1. Loss Factors – Boundary Perturbation Method (BP) *vs.* TBCI

| No. | Geometry | | $\sigma$ | $k_{\parallel}^{m=0}$ $(V/pC)$ | | $k_{\perp}^{m=1}$ $(V/pC \cdot m)$ | |
|---|---|---|---|---|---|---|---|
| | $b_0$ / $\epsilon$ / $g$ $(cm)$ | $\theta$ | $(cm)$ | BP | TBCI | BP | TBCI |
| 1 | 1.8 / 0.2 / 0.4 | 45° | 1.75 | 0.83E-4 | 1.3E-4 | 2.35 | 2.33 |
| 2 | 1.8 / 0.2 / 0.4 | 45° | 0.5 | 0.083 | 0.094 | 7.55 | 7.82 |
| 3 | 1.6 / 0.4 / 0.8 | 45° | 0.5 | 0.31 | 0.36 | 30.3 | 31.6 |
| 4 | 1.2 / 0.8 / 1.6 | 45° | 0.5 | 0.54 | 0.87 | 176 | 263 |
| 5 | 1.2 / 0.8 / 20 | 5° | 1.76 | 1E-8 | 1E-5 | 5.0 | < 11 |
| 6 | 1.2 / 0.8 / 20 | 5° | 0.58 | 2E-3 | 4E-3 | 15 | < 49 |

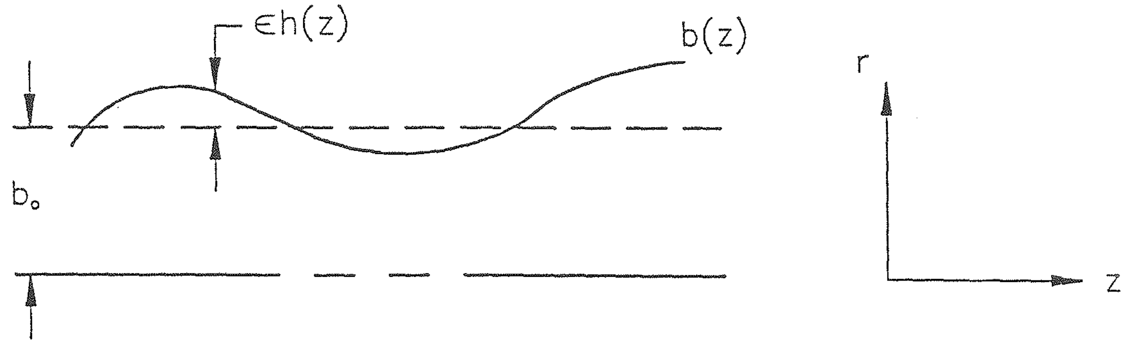


Fig. 1. The unperturbed boundary $b_0$ and the perturbed one $b(z)$.

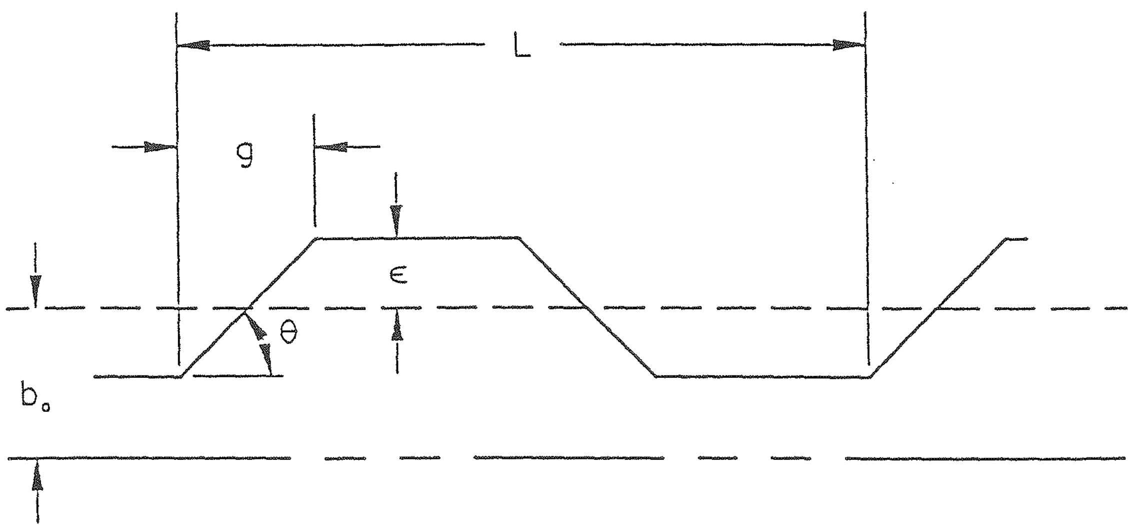


Fig. 2. A periodic structure which consists of a series of transitions between beam chamber and insertion device section.

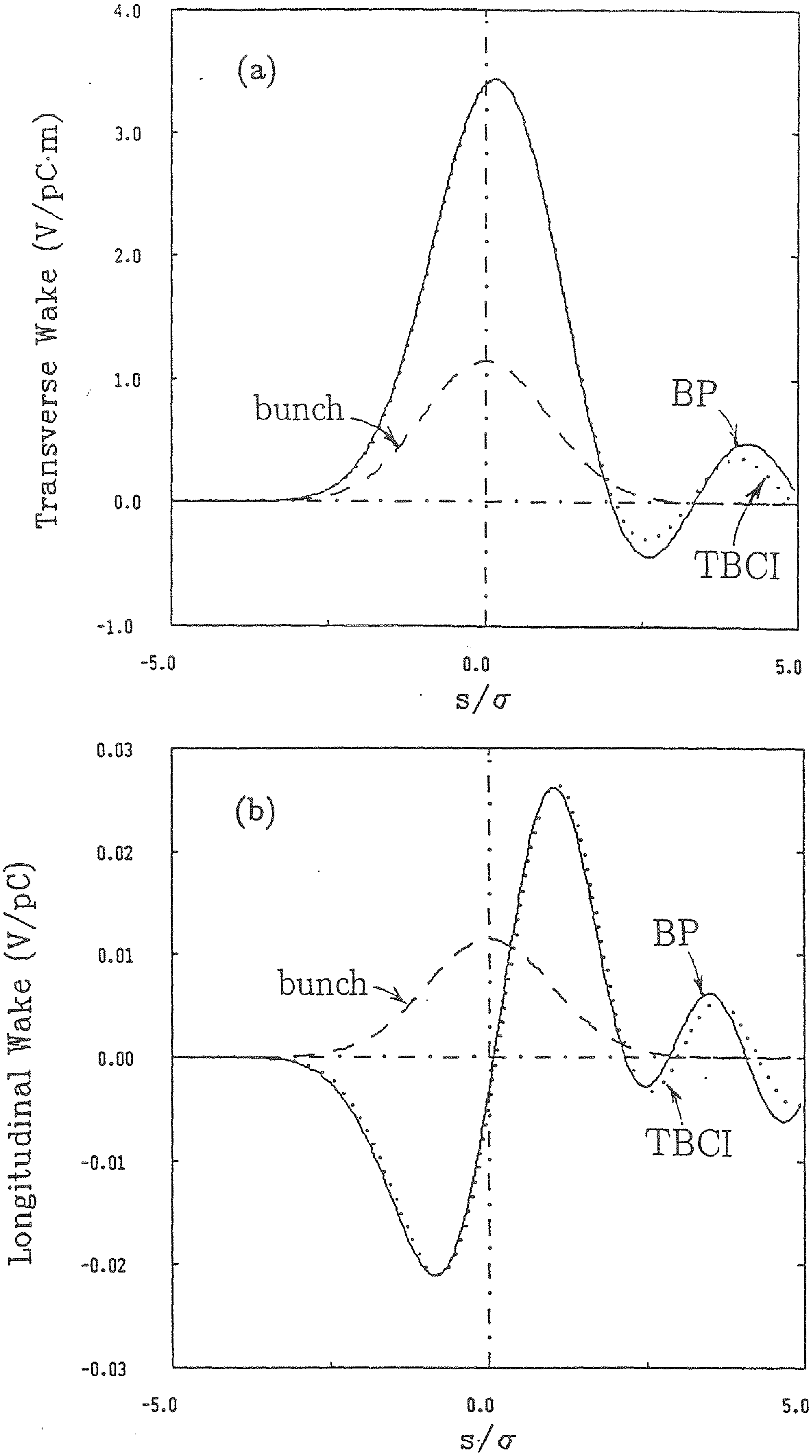


Fig. 3. The wakefields of a Gaussian bunch calculated by BP (solid line) and by TBCI (dotted line) for the parameter list (16) in the text. The left side is the bunch head. (a) Transverse and (b) Longitudinal.

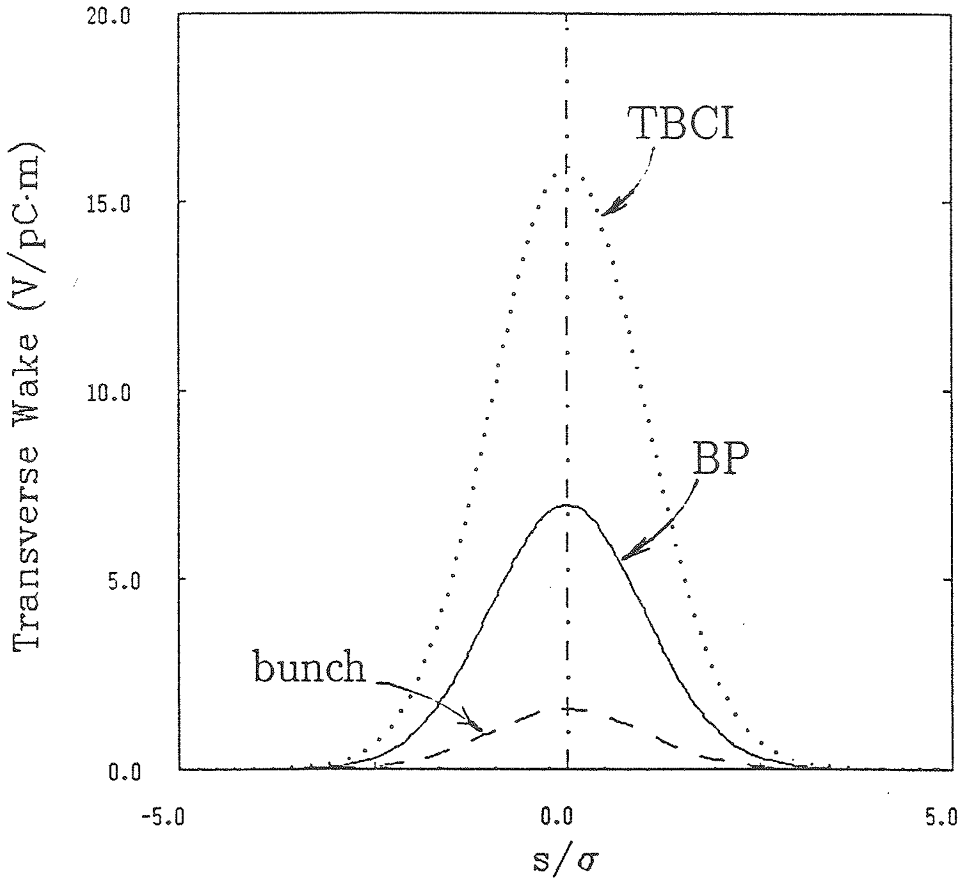


Fig. 4. The wakefields of a Gaussian bunch calculated by BP (solid line) and by TBCI (dotted line) for the parameter list (17) in the text. The left side is the bunch head.